WI4014TU – Numerical Analysis for PDE's

Lecture 2
Classification and Theoretical Analysis of PDE's

Classification of PDE's

Second-order PDE's with two variables

The second-order partial differential equation with two independent variables x and y is a relation between the unknown function $u(x, y)$ and its partial derivatives up to and including the second-order:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0$$

Linear second-order PDE's

A second-order PDE is **linear with respect to the highest-order derivatives** if it has the form:

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + F_1\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

where the coefficients a_{11}, a_{12}, and a_{22} are functions of x and y only. The equation is **quasi-linear** if these coefficients are also functions of u, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$. The equation is **linear** if it has the form:

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + cu + f = 0$$

and a_{11}, a_{12}, a_{22}, b_1, b_2, c, f are functions of x and y only. If these functions are constants, then the equation is **linear with constant coefficients**.

Matrix of coefficients

Consider the linear (in highest derivatives) case:

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

This we can also write as

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} + F = 0$$

or simply as

$$\nabla \cdot \mathbb{A} \nabla u + F = 0,$$

where $\mathbb{A}$ is the matrix of coefficients.

Change of variables

Since $\mathbb{A}$ is symmetric, $\mathbb{A} = \mathbb{Q}\Lambda\mathbb{Q}^T$ and

$$\nabla \cdot \mathbb{Q}\Lambda\mathbb{Q}^T \nabla u + F = 0,$$
$$\tilde{\nabla} \cdot \Lambda\tilde{\nabla} u + F = 0,$$

where Λ is the diagonal matrix of eigenvalues of $\mathbb{A}$ and $\tilde{\nabla}$ is the gradient operator with respect to new variables ξ and η. The PDE becomes

$$\lambda_1 \frac{\partial^2 u}{\partial \xi^2} + \lambda_2 \frac{\partial^2 u}{\partial \eta^2} + F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$

Hint, $u(x(\xi, \eta), y(\xi, \eta))$:

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} = \frac{\partial u}{\partial x} q_{11} + \frac{\partial u}{\partial y} q_{12}$$

Classification of second-order PDE's

Eigenvalues λ_1 and λ_2 of the symmetric $\mathbb{A}$ are real. There are three distinct possibilities:

1. λ_1 and λ_2 both nonzero and have the same sign. Equation is **elliptic**, e.g.:
$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$
2. λ_1 and λ_2 both nonzero and have different signs. Equation is **hyperbolic**, e.g.:
$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$
3. One of λ_1 and λ_2 is zero. Equation is **parabolic**, e.g.:
$$\frac{\partial^2 u}{\partial \xi^2} + F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0$$

How to classify a second-order PDE

Reduce the original PDE to the form

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Expressing the eigenvalues of the coefficient matrix $\mathbb{A}$ via the coefficients a_{11}, a_{12}, and a_{22}, we find:

1. Equation is **elliptic** if $a_{12}^2 - a_{11}a_{22} < 0$
2. Equation is **hyperbolic** if $a_{12}^2 - a_{11}a_{22} > 0$
3. Equation is **parabolic** if $a_{12}^2 - a_{11}a_{22} = 0$

The class of PDE may change throughout time and space, since coefficients may depend on t, x, and y.

Example: Poisson's equation

Poisson's equation with differentiable $k(x, y)$ is elliptic:

$$
\begin{aligned}
&-\nabla \cdot (k \nabla u) = f \\
&-k \nabla \cdot \nabla u - (\nabla k) \cdot (\nabla u) = f \\
&-k \frac{\partial^2 u}{\partial x^2} - k \frac{\partial^2 u}{\partial y^2} - \frac{\partial k}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial k}{\partial y} \frac{\partial u}{\partial y} = f
\end{aligned}
$$

Two variables x and y. Usually, $k(x, y) > 0$.

Poisson's and Laplace's equations are elliptic in three dimensions as well: signs in front of all three second-order derivatives are the same.

Example: wave equation

Wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f$$

There are three variables in $u(x, y, t)$: x, y, and t. Same eigenvalue analysis as in $2D$ but with a 3×3 matrix of coefficients. Similar conclusions about the signs of coefficients. Coefficients in front of the second-order spatial derivatives have different sign compared to the second-order time derivative. Hence, wave equation is hyperbolic.

Wave equation is hyperbolic in three dimensions as well:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = f$$

Four variables: x, y, z, and t. Sign in front of the second-order derivative with respect to time t is different from the rest.

Example: advection equation

The advection equation is of the first-order:

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u = f$$

but is also considered to be hyperbolic. Similar to the wave equation.

Example: diffusion equation

Diffusion equation:

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = f$$

There are two variables x and t. Coefficient in front of the second-order derivative with respect to t is zero. Hence, diffusion equation is parabolic.

Diffusion equation is parabolic in two and three dimensions as well:

$$\frac{\partial u}{\partial t} - \nabla \cdot k\nabla u = f$$

Four variables: x, y, z, and t. Second derivative in t is missing.

Example: convection-diffusion equation

Convection-diffusion equation

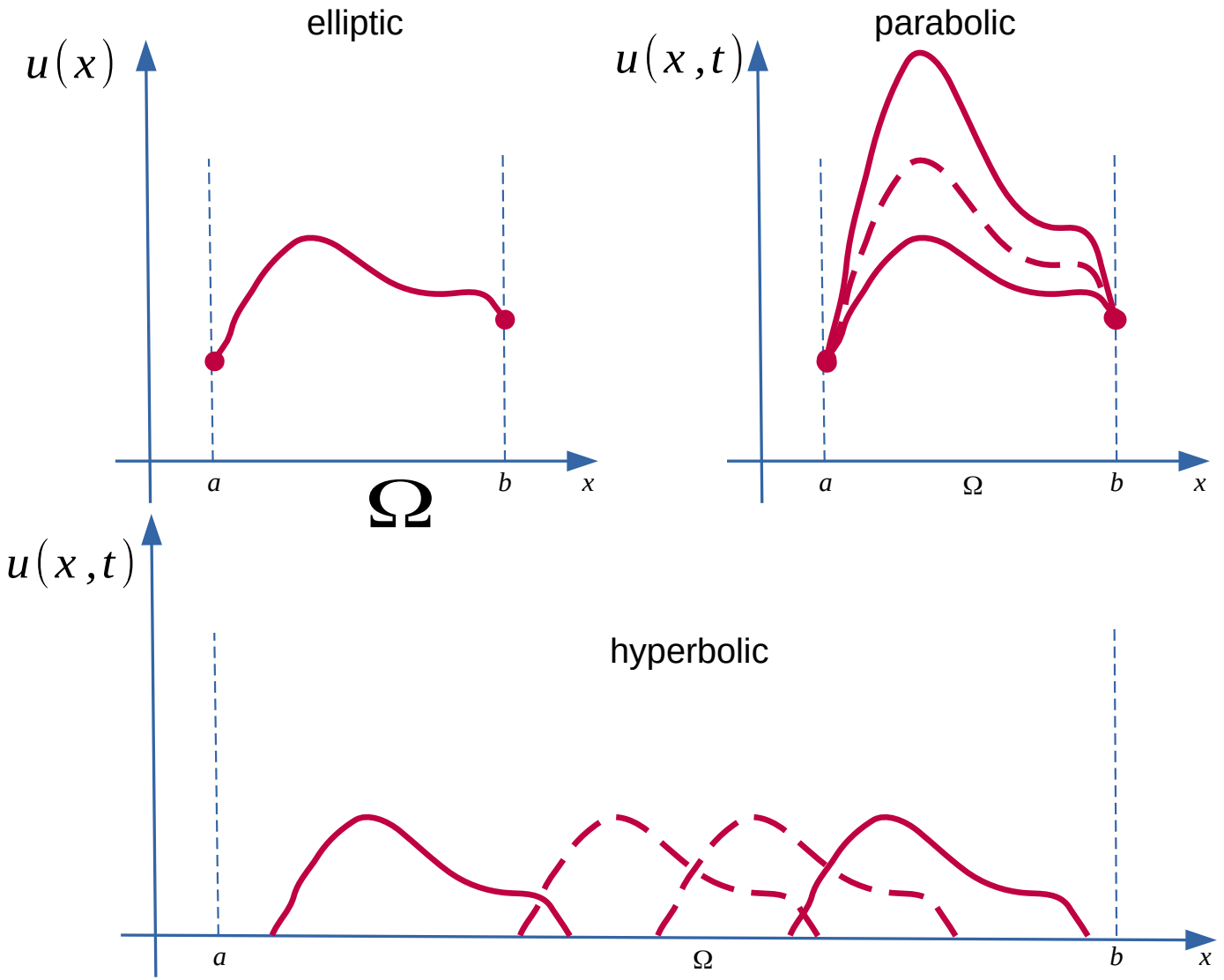
$$\frac{\partial u}{\partial t} - \nabla \cdot k \nabla u + \boldsymbol{v} \cdot \nabla u = f$$

is formally parabolic if $k > 0$. However, if $k \ll |\boldsymbol{v}|$, then convection *dominates* over diffusion, and the equation is almost hyperbolic. This is called a **singular perturbation** of the problem.

Why do we classify equations?

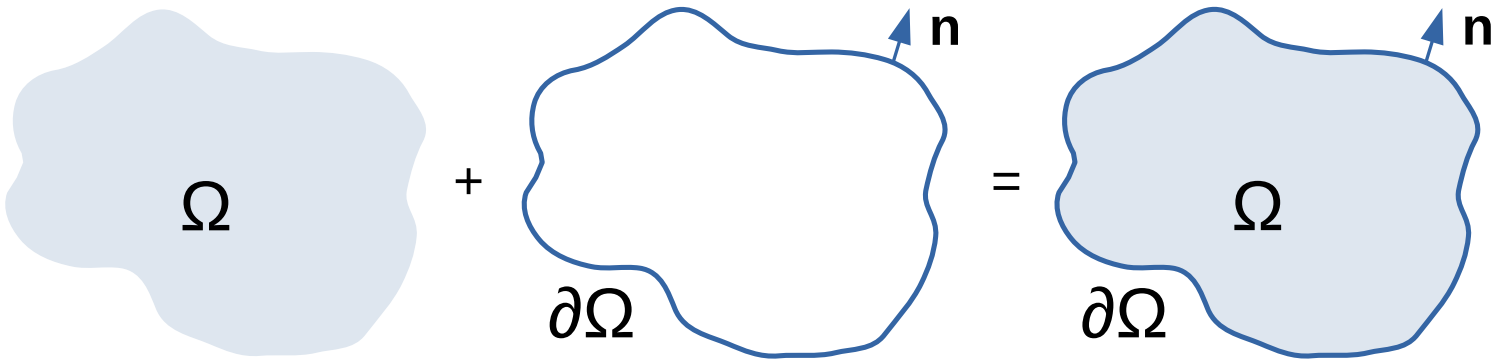
- Elliptic equations describe steady-state problems
- Parabolic equations describe very smooth dynamics (diffusion) towards steady state
- Hyperbolic equations describe waves, i.e., their solutions move through space, often preserving their shape

Typical behavior



Boundary and initial conditions

Domain Ω, boundary surface $\partial\Omega$, normal vector $\boldsymbol{n}$



$\Omega \cup \partial\Omega = \overline{\Omega}$

Boundary conditions for Poisson's equation

Let $u(\boldsymbol{x})$ satisfy

$$-\nabla \cdot k \nabla u = f$$

for all $\boldsymbol{x} \in \Omega \subset \mathbb{R}^p$, $p = 1, 2, 3$. Let $\partial\Omega$ be the boundary of Ω with the outwards normal vector $\boldsymbol{n}$. The function u may be required to satisfy the following common boundary conditions (BC):

1. **Dirichlet BC:** $u(\boldsymbol{x}) = g_0(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$
2. **Neumann BC:** $k(\boldsymbol{x}) D_{\boldsymbol{n}} u(\boldsymbol{x}) = g_1(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$
3. **Robin (mixed) BC:** $k(\boldsymbol{x}) D_{\boldsymbol{n}} u(\boldsymbol{x}) + \sigma u(\boldsymbol{x}) = g_2(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$

where $D_{\boldsymbol{n}} u$, also denoted as $\partial u / \partial \boldsymbol{n}$ is the directional derivative of u, and $g_{0,1,2}(\boldsymbol{x})$ are given functions.

The boundary $\partial\Omega$ may be split into several disjoint parts. And u may be required to satisfy different BC's at different parts of $\partial\Omega$.

Example: electrostatic potential

The electrostatic potential ϕ is maintained at the value of 5 V at the metallic piece $\partial\Omega_1$ of the boundary $\partial\Omega$ surrounding the domain Ω. The normal component of the electric field $\mathbf{E} = -\nabla\phi$ is zero at the rest of the boundary $\partial\Omega_2$. The **boundary value problem** for ϕ is:

$$\begin{aligned} -\Delta\phi(\boldsymbol{x}) &= 0, \quad \boldsymbol{x} \in \Omega \\ \phi(\boldsymbol{x}) &= 5, \quad \boldsymbol{x} \in \partial\Omega_1 \\ \mathbf{n} \cdot \nabla\phi(\boldsymbol{x}) &= D_{\boldsymbol{n}}\phi(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega_2 \end{aligned}$$

Boundary and initial conditions for diffusion equation

Let $u(\boldsymbol{x}, t)$ satisfy

$$\frac{\partial u}{\partial t} - \nabla \cdot k \nabla u = f$$

for all $\boldsymbol{x} \in \Omega \subset \mathbb{R}^p$, $p = 1, 2, 3$ and all $t \geq 0$. The following BC's are common:

1. Dirichlet BC: $u(\boldsymbol{x}, t) = g_0(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \partial\Omega$, $t \geq 0$
2. Neumann BC: $k(\boldsymbol{x}) D_{\boldsymbol{n}} u(\boldsymbol{x}, t) = g_1(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \partial\Omega$, $t \geq 0$
3. Robin BC: $k(\boldsymbol{x}) D_{\boldsymbol{n}} u(\boldsymbol{x}, t) + \sigma u(\boldsymbol{x}, t) = g_2(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \partial\Omega$, $t \geq 0$
4. **Initial condition:** $u(\boldsymbol{x}, t) = u_0(\boldsymbol{x})$ at $t = 0$

where $g_{0,1,2}(\boldsymbol{x}, t)$ and $u_0(\boldsymbol{x})$ are given functions.

Example: heat flow

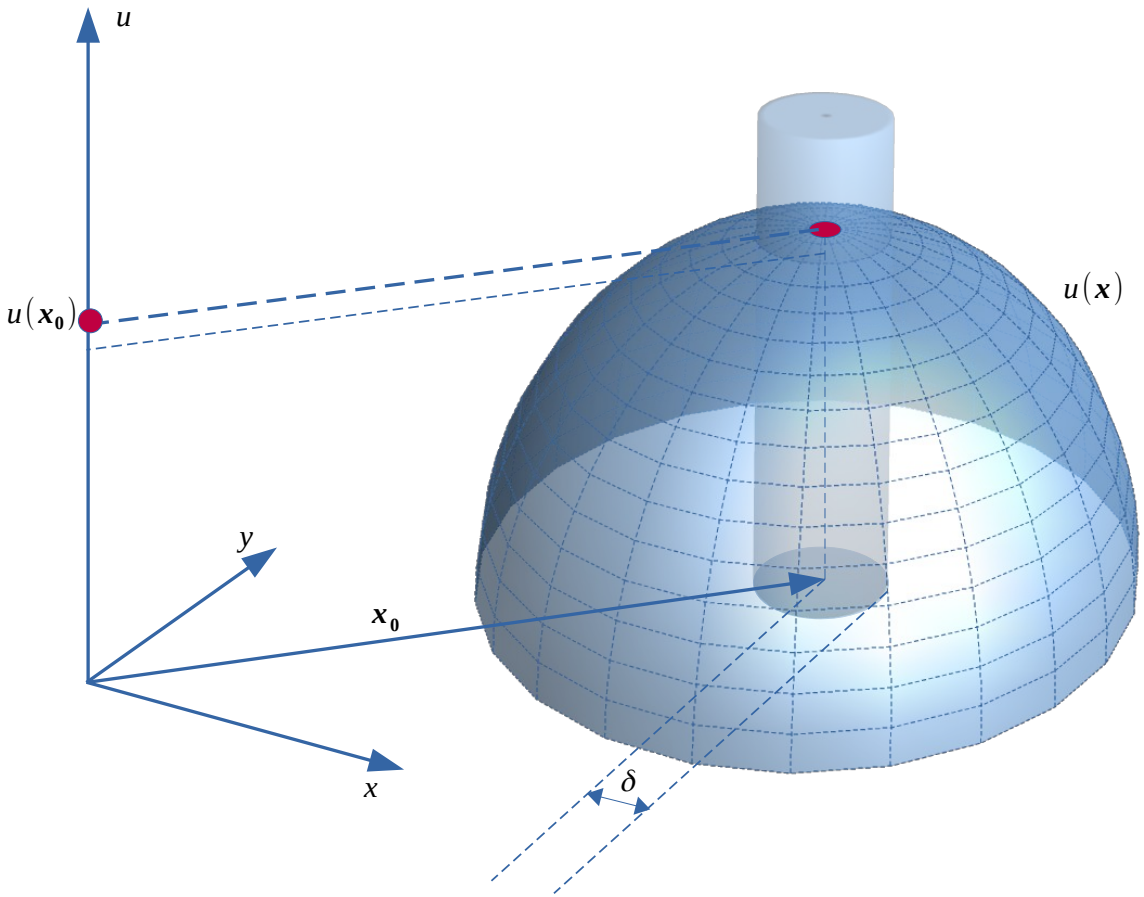
Find the temperature distribution $T(\boldsymbol{x}, t)$ inside a metallic rod with the thermal conductivity k. Initial temperature of the rod is T_1. The temperature at one end of the rod remains T_1 at all times. The heat flux density through the other end of the rod is $q(t)$, $t \geq 0$. The rest of the rod surface exchanges heat with the environment at temperature T_0 by thermal radiation.

$$\begin{aligned}
\frac{\partial T}{\partial t} - k\Delta T &= 0, \quad \boldsymbol{x} \in \Omega, \quad t > 0 \\
T(\boldsymbol{x}, 0) &= T_1, \quad \boldsymbol{x} \in \Omega \\
T(\boldsymbol{x}, t) &= T_1, \quad \boldsymbol{x} \in \partial\Omega_1, \quad t \geq 0 \\
kD_{\mathbf{n}} T(\boldsymbol{x}, t) &= -q(t), \quad \boldsymbol{x} \in \partial\Omega_2, \quad t \geq 0 \\
kD_{\mathbf{n}} T(\boldsymbol{x}, t) + \lambda\left(T(\boldsymbol{x}, t) - T_0\right) &= 0, \quad \boldsymbol{x} \in \partial\Omega_3, \quad t \geq 0
\end{aligned}$$

where $\partial\Omega_1$ is the surface with prescribed temperature, $\partial\Omega_2$ is the surface with prescribed flux density, and $\partial\Omega_3$ is the surface where the heat is exchanged with the environment.

Maximum Principle and Uniqueness of Solutions

Maximum of a function



What happens at and around a maximum

Definition (maximum): A function $u(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$ has a maximum at $\boldsymbol{x}_0 \in \Omega \subset \mathbb{R}^2$ if there exists $\delta > 0$ such that $u(\boldsymbol{x}_0) \geq u(\boldsymbol{x})$ for all $\boldsymbol{x}$, $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$.

Definition (Hessian): The hessian matrix of $u(\boldsymbol{x})$ at $\boldsymbol{x}_0 \in \mathbb{R}^2$ is defined as

$$H(\boldsymbol{x}_0) = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2}(\boldsymbol{x}_0) & \frac{\partial^2 u}{\partial x \partial y}(\boldsymbol{x}_0) \\ \frac{\partial^2 u}{\partial y \partial x}(\boldsymbol{x}_0) & \frac{\partial^2 u}{\partial y^2}(\boldsymbol{x}_0) \end{bmatrix}$$

Necessary condition: For sufficiently smooth functions, such as $u(\boldsymbol{x}) \in C^2(\Omega)$, if $u(\boldsymbol{x})$ has a maximum at $\boldsymbol{x}_0$, then $\nabla u(\boldsymbol{x}_0) = \boldsymbol{0}$.

Theorem: If a function $u(\boldsymbol{x}) \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^2$ has a maximum at $\boldsymbol{x}_0$, then there exists $\delta > 0$ such that for all $\boldsymbol{x}$, $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$, the Hessian matrix of $u(\boldsymbol{x})$ at $\boldsymbol{x}_0$ is negative semi-definite, i.e. $(\boldsymbol{x} - \boldsymbol{x}_0)^T H(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0$, for all $\boldsymbol{x}$, $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$

From Hessian to Laplacian

Observation: If $H(\boldsymbol{x}_0)$ is negative semi-definite, then $-\Delta u(\boldsymbol{x}_0) \geq 0$.

Corollary: Assume that $u(\boldsymbol{x})$ is defined and sufficiently smooth in the neighborhood of $\boldsymbol{x}_0$. Then,

- If $u(\boldsymbol{x})$ has a maximum in $\boldsymbol{x}_0$, then $\nabla u(\boldsymbol{x}_0) = \mathbf{0}$ and $-\Delta u(\boldsymbol{x}_0) \geq 0$
- If $u(\boldsymbol{x})$ has a minimum in $\boldsymbol{x}_0$, then $\nabla u(\boldsymbol{x}_0) = \mathbf{0}$ and $-\Delta u(\boldsymbol{x}_0) \leq 0$

Harmonic functions

Definition: A function $u(\boldsymbol{x})$ satisfying:

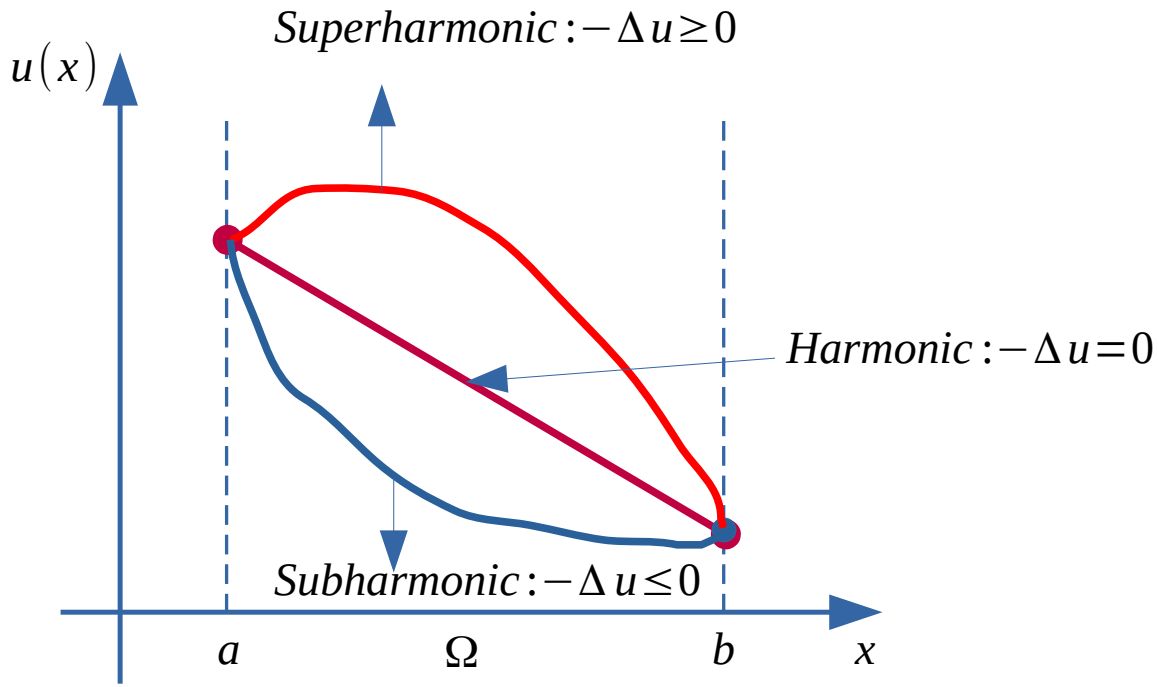
- $-\Delta u = 0$ for $\boldsymbol{x} \in \Omega$ is called **harmonic** on Ω
- $-\Delta u \leq 0$ for $\boldsymbol{x} \in \Omega$ is called **subharmonic** on Ω
- $-\Delta u \geq 0$ for $\boldsymbol{x} \in \Omega$ is called **superharmonic** on Ω

Maximum Principle

Theorem (Maximum Principle): Let Ω be an open bounded domain with boundary $\partial\Omega$ and closure $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is *subharmonic* on Ω. Then,

1. $u(\boldsymbol{x})$ takes its maximum at the boundary $\partial\Omega$, hence, $\max_{\boldsymbol{x}\in\overline{\Omega}} u = \max_{\boldsymbol{x}\in\partial\Omega} u$
2. Furthermore, if Ω is connected and if there is an internal point $\boldsymbol{x}_0$ where $u(\boldsymbol{x})$ reaches its maximum $u(\boldsymbol{x}_0) = \max_{\boldsymbol{x}\in\overline{\Omega}} u$, then $u(\boldsymbol{x})$ is constant on $\overline{\Omega}$, i.e., $u(\boldsymbol{x}) = u(\boldsymbol{x}_0)$ on $\overline{\Omega}$.

A similar (Minimum Principle) theorem holds for *superharmonic* functions with maximum replaced by the minimum.


Superharmonic : $-\Delta u \geq 0$
$u(x)$
Harmonic : $-\Delta u = 0$
Subharmonic : $-\Delta u \leq 0$
a
Ω
b
x

Uniqueness of the solution with Dirichlet BC's

Theorem: Laplace's equation $-\Delta u = 0$ on Ω with a *homogeneous* Dirichlet boundary condition $u(\boldsymbol{x}) = 0$, $\boldsymbol{x} \in \partial\Omega$, has only the trivial solution $u(\boldsymbol{x}) = 0$ on Ω.

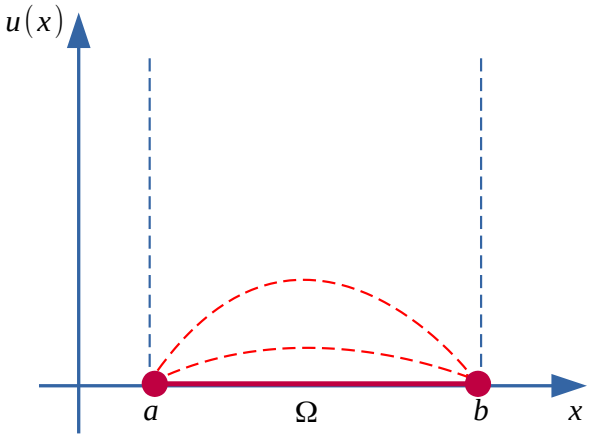
Theorem (uniqueness, Dirichlet): Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$, and suppose that $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ satisfies

$$\begin{aligned} -\Delta u &= f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \\ u &= g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega \end{aligned}$$

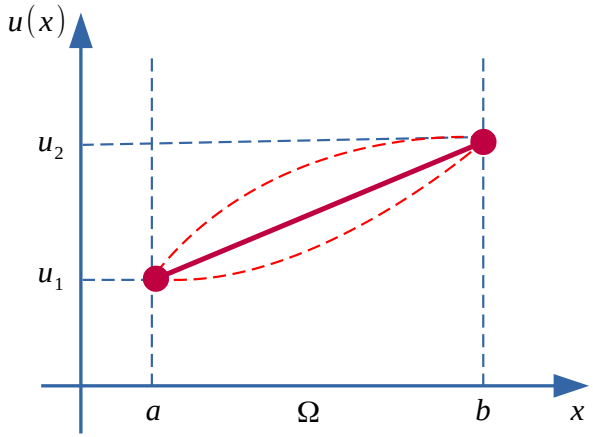
Then there exists at most one such solution $u(\boldsymbol{x})$.

Uniqueness of the solution with Dirichlet BC's

$-u''=0, u(a)=0, u(b)=0$



$-u''=0, u(a)=u_1, u(b)=u_2$



Uniqueness of the solution with Robin BC's

Theorem (uniqueness, Robin): Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$, and suppose that $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ satisfies

$$-\Delta u = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$$
$$\sigma u + D_{\boldsymbol{n}} u = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega$$

with $\sigma > 0$. Then there exists at most one such solution $u(\boldsymbol{x})$.

Non-uniqueness of the solution with Neumann BC's

Theorem (non-uniqueness, Neumann): Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$, and suppose that $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ satisfies

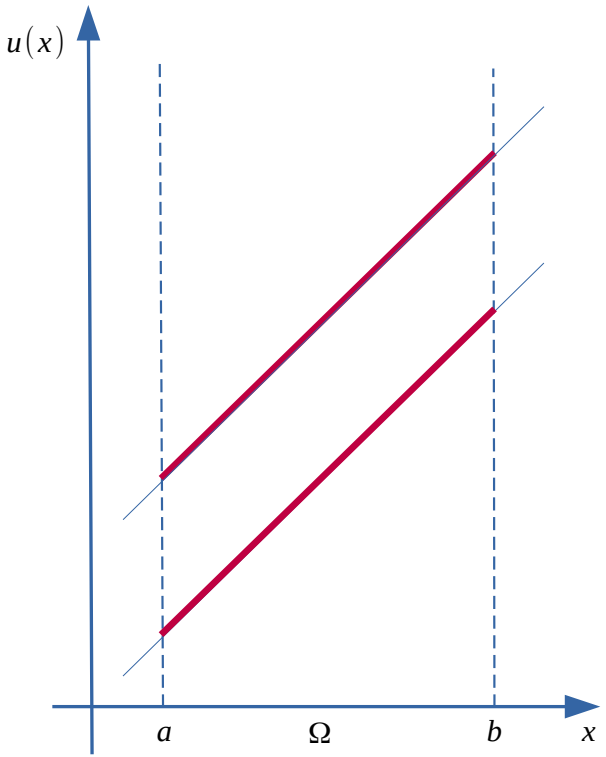
$$\begin{aligned} -\nabla \cdot k\nabla u &= f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \\ -kD_{\boldsymbol{n}}u &= g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega \end{aligned}$$

Then the following *compatibility condition* must be satisfied:

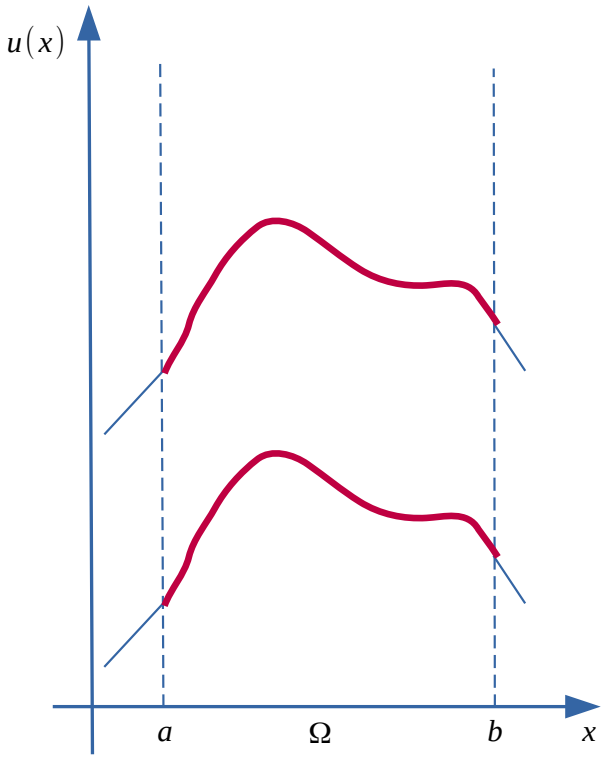
$$\int_{\partial\Omega} g \, ds = \iint_{\Omega} f \, dA$$

and the solution $u(\boldsymbol{x})$ is determined up to an additive constant only, i.e., $u(\boldsymbol{x}) + c$ is also a solution.

Non-uniqueness of the solution with Neumann BC's



$-u''=0, u'(a)=1, -u'(b)=-1$



$-u''=f, u'(a)=g_1, -u'(b)=g_2$

Non-negativity

Theorem (non-negativity): Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$, and suppose that $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ satisfies

$$\begin{aligned} -\Delta u &\geq 0, \quad \boldsymbol{x} \in \Omega \\ u &= 0, \quad \boldsymbol{x} \in \partial\Omega \end{aligned}$$

Then $u(\boldsymbol{x}) \geq 0$ on Ω.

- This allows figuring out the sign of u, if $-\Delta u = f$, $\boldsymbol{x} \in \Omega$; $u = 0$, $\boldsymbol{x} \in \partial\Omega$, and the sign of f is given.
- What if $-\Delta u = f$, $\boldsymbol{x} \in \Omega$; $u = g$, $\boldsymbol{x} \in \partial\Omega$, and the signs of f and g are given?

Stability with respect to changes in BC

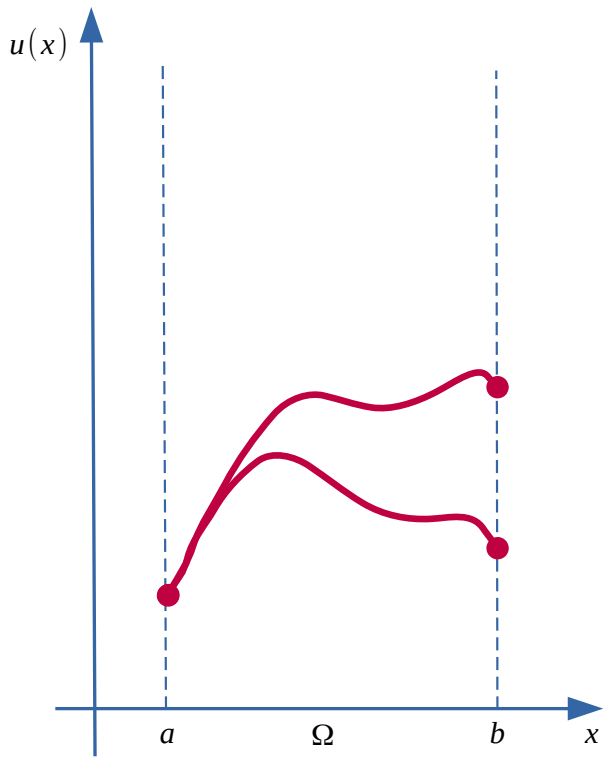
Theorem (stability to changes in BC): Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$, and suppose that $u_1, u_2 \in C^2(\Omega) \cap C^1(\partial\Omega)$ satisfy

$$\begin{cases} -\Delta u_1 = f, & \boldsymbol{x} \in \Omega; \\ u_1 = g_1, & \boldsymbol{x} \in \partial\Omega; \end{cases}$$

$$\begin{cases} -\Delta u_2 = f, & \boldsymbol{x} \in \Omega; \\ u_2 = g_2, & \boldsymbol{x} \in \partial\Omega; \end{cases}$$

Then, $|u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})| \leq \max_{\boldsymbol{x}\in\partial\Omega} |g_1(\boldsymbol{x}) - g_2(\boldsymbol{x})|$, for all $\boldsymbol{x} \in \Omega$.

Stability of the solution



$$-u''=f, u(a)=g_1, u(b)=g_2 \rightarrow u(b)=g_3$$