WI4014TU – Numerical Analysis for PDE's

Lecture 9
Finite Element Method: Ritz method for self-adjoint operators, Fourier basis, linear Lagrangian basis, 1D Poisson's equation with Dirichlet's and mixed boundary conditions

Self-adjoint operators: Ritz Method

PDE's obtained from minimization of functionals

Previously we have shown that:

- Solution of a PDE with a self-adjoint differential operator minimizes a functional:

$$\mathcal{L}(u) = f$$

$$L(u) = \int_{\Omega} \left[\frac{1}{2} u\mathcal{L}(u) - fu \right] d\Omega$$

- Minimum of a functional over a function space $U(\Omega)$ satisfies Poisson's equation.
- Ritz Method solves the minimization problem numerically. Works with PDE's that can be reduced to minimization problems, i.e., with self-adjoint differential operators.
- Some ideas of Ritz Method are present in other Finite Element Methods

Poisson's equation as a minimization problem

- Ω – open bounded domain in $\mathbb{R}^n$, $n = 1, 2, 3$; $\partial\Omega = \partial\Omega_{\mathrm{D}} \cup \partial\Omega_{\mathrm{N}}$
- $k(\boldsymbol{x}) > 0$; $f(\boldsymbol{x}) \in C(\Omega)$; $g(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega_{\mathrm{D}}$; $h(\boldsymbol{x}) \in C(\partial\Omega_{\mathrm{N}})$
- $U(\Omega) = \{u \,|\, \nabla \cdot (k\nabla u) \in C(\Omega);\ u(\boldsymbol{x}) = g(\boldsymbol{x}), \boldsymbol{x} \in \partial\Omega_{\mathrm{D}}\}$

In Lecture 8 we showed that to find $u \in U(\Omega)$ that minimizes the functional:

$$F(u) = \int_\Omega \left[\frac{1}{2}\nabla u \cdot (k\nabla u) - fu\right] d\Omega + \int_{\partial\Omega_{\mathrm{N}}} hu\, d\Gamma, \quad u \in U(\Omega)$$

it is sufficient to solve the Poisson equation

$$-\nabla \cdot (k\nabla u) = f, \quad \boldsymbol{x} \in \Omega$$

with BC's:

$$u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_{\mathrm{D}}$$
$$-k(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \boldsymbol{n} = h(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega_{\mathrm{N}}$$

Ritz Method: minimization over a subspace

What if try to minimize the functional

$$F(u) = \int_\Omega \left[\frac{1}{2} \nabla u \cdot (k \nabla u) - fu \right] d\Omega + \int_{\partial\Omega_N} hu \, d\Gamma$$

Ritz Method:

- Choose a *finite-dimensional subspace* $U_h(\Omega) \subset U(\Omega)$
- Find $u_h \in U_h(\Omega)$ that minimizes $F(u)$ on $U_h(\Omega)$:
  1. Introduce suitable *basis* $\{u_i(\boldsymbol{x})\}$ for $U_h(\Omega)$
  2. Represent the solution as: $u_h(\boldsymbol{x}) = \sum_i a_i u_i(\boldsymbol{x})$
  3. Find *coefficients* $\{a_i\}$ that minimize $F(u_h)$

Minimization with respect to coefficients: Ritz's equations

To find extremum of $F(u_h)$ with respect to coefficients $\{a_j\}$, solve a finite number of Ritz's equations

$$\frac{\partial F(u_h)}{\partial a_1} = 0$$
$$\frac{\partial F(u_h)}{\partial a_2} = 0$$
$$\dots$$
$$\frac{\partial F(u_h)}{\partial a_n} = 0$$

Short notation: $\frac{\partial F(u_h)}{\partial a_j} = 0, \quad j = 1, \dots, n$

Example: 1D Poisson with homogeneous Dirichlet BC's

Equation to be solved:

$$-u'' = f, \quad x \in (a, b)$$
$$u(a) = u(b) = 0$$

Corresponding functional to be minimized:

$$F(u) = \int_a^b \left[\frac{1}{2}(u')^2 - fu \right] dx$$

Function space:

$$U(\Omega) = \{u \,|\, u'' \in C(\Omega);\ u(a) = u(b) = 0\} = C_0^2(\Omega)$$

Introducting finite-dimensional subspace $U_h(\Omega)$

- A finite-dimensional subspace $U_h(\Omega) \subset U(\Omega)$ of dimension $\dim(U_h) = n$ is defined by a set of basis functions $\{u_i(x), i = 1, \ldots, n\}$. To define a subspace, simply define its basis.
- If you have non-zero Dirichlet BC's, then you define a full set of basis functions satisfying zero BC's and a minimum number of additional basis functions that allow to satisfy nonzero BC's.
- Any $u_h \in U_h(\Omega)$ can be represented as a linear combination of basis functions

$$u_h(x) = \sum_{i=1}^{n} a_i u_i(x)$$

where a_i are the *expansion coefficients*.

- On $U_h(\Omega)$ the functional becomes

$$F(u_h) = \int_a^b \left[\frac{1}{2} \left(\sum_{i=1}^{n} a_i u_i' \right)^2 - f \sum_{i=1}^{n} a_i u_i \right] dx$$

Minimizing $F(u_h)$ with respect to coefficients

To find coefficients, solve Ritz's equations

$$\frac{\partial F(u_h)}{\partial a_j} = 0, \quad j = 1, \ldots, n$$

$$\frac{\partial}{\partial a_j} \int_a^b \left[\frac{1}{2} \left(\sum_{i=1}^n a_i u_i' \right)^2 - f \sum_{i=1}^n a_i u_i \right] dx$$

$$= \int_a^b \left[\frac{1}{2} 2 \sum_{i=1}^n a_i u_i' \frac{\partial}{\partial a_j} \sum_{i=1}^n a_i u_i' - f \frac{\partial}{\partial a_j} \sum_{i=1}^n a_i u_i \right] dx$$

$$= \int_a^b \left[\sum_{i=1}^n a_i u_i' u_j' - f u_j \right] dx = \sum_{i=1}^n a_i \int_a^b u_i' u_j' \, dx - \int_a^b f u_j \, dx = 0$$

Ritz's equations as a linear algebraic problem

$$\sum_{i=1}^{n} a_i \int_a^b u_i' u_j' \, dx = \int_a^b f u_j \, dx, \quad j = 1, \ldots, n$$

This is a linear system of equations

$$\boxed{S\boldsymbol{a} = \boldsymbol{f}}$$

$$s_{ij} = \int_a^b u_i' u_j' \, dx, \quad \boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} \int_a^b f u_1 \, dx \\ \int_a^b f u_2 \, dx \\ \vdots \\ \int_a^b f u_n \, dx, \end{bmatrix}$$

where $\boldsymbol{a}, \boldsymbol{f} \in \mathbb{R}^n$ and $S \in \mathbb{R}^{n \times n}$ is the *Gramian matrix* of functions $\{u_i'\}$ also called *stiffness matrix*.

Global basis functions: Fourier basis

- Let $\Omega = (0,1)$, $u(0) = u(1) = 0$
- $U(\Omega) = \{u \,|\, u'' \in C(\Omega);\ u(0) = u(1) = 0\} = C_0^2(\Omega)$
- We define subspace $U_h(\Omega) \subset U(\Omega)$ by choosing its basis as $u_i(x) = \sin(i\pi x)$, $i = 1, \ldots, n$. In other words: $U_h(\Omega) = \{u \,|\, u(x) = \sum_{i=1}^{n} a_i \sin(i\pi x), x \in \overline{\Omega}\}$
- Basis functions are *global* but *orthogonal*

$$s_{ij} = \int_a^b u_i' u_j' \, dx = ij\pi^2 \int_0^1 \cos(i\pi x)\cos(j\pi x)\, dx = \begin{cases} i^2\pi^2/2, & j = i \\ 0, & j \neq i \end{cases}$$

$$f_i = \int_0^1 f(x)\sin(i\pi x)\, dx$$

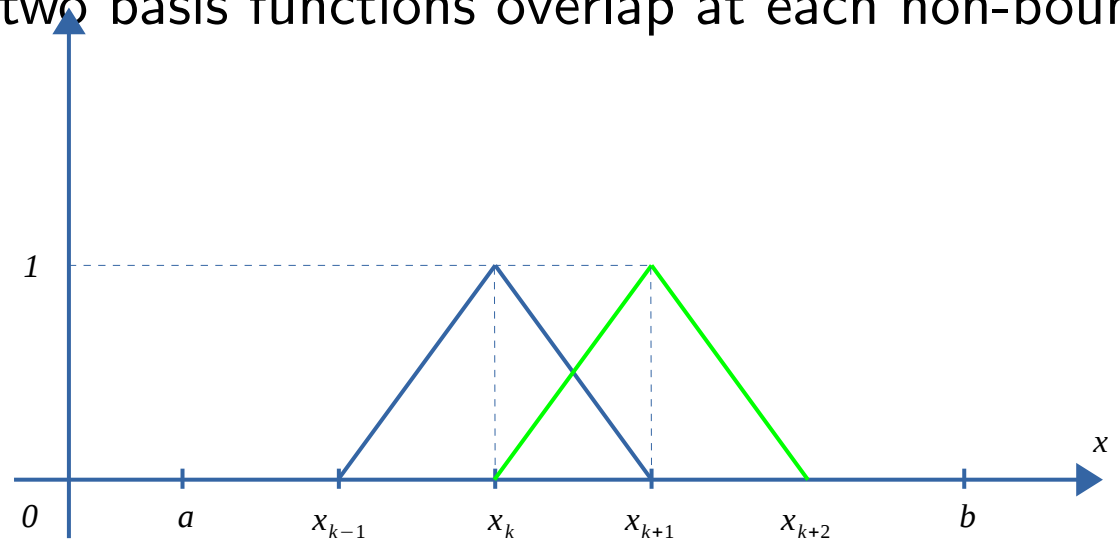
- Since S is diagonal, the solution is explicit:

$$u_h(x) = \sum_{i=1}^{n} a_i \sin(i\pi x), \quad a_i = \frac{f_i}{s_{ii}} = \frac{2}{i^2\pi^2} \int_0^1 f(x)\sin(i\pi x)\, dx$$

Local basis functions: Finite Elements

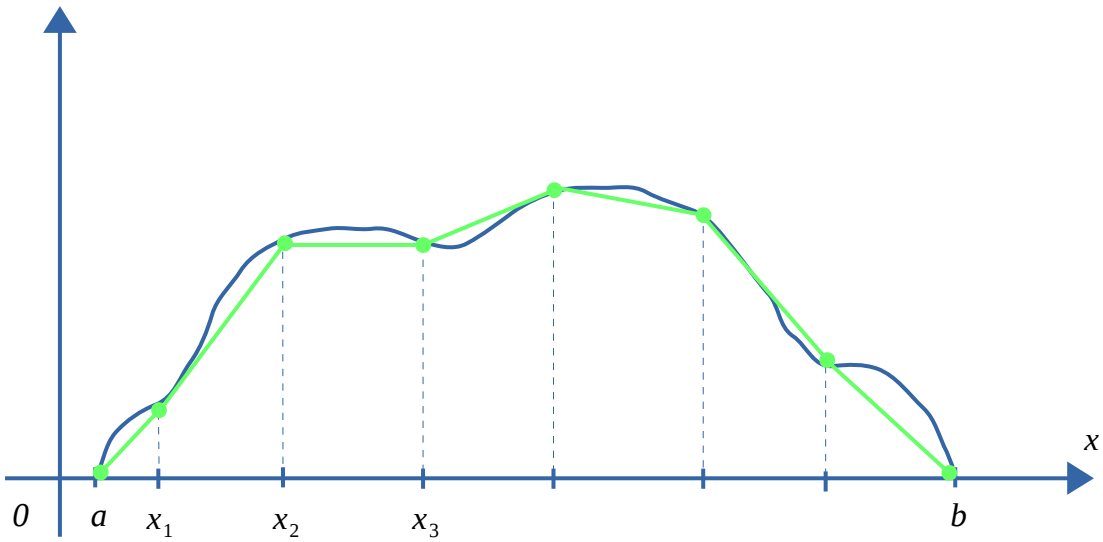
$$u_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}} & \text{if } x_{k-1} \le x \le x_k \\ \frac{x_{k+1} - x}{x_{k+1} - x_k} & \text{if } x_k \le x \le x_{k+1} \\ 0 \text{ otherwise} \end{cases}$$

- In this example we choose *linear* Lagrangian functions
- *Finite elements* are the intervals $[x_{k-1}, x_k]$, $[x_k, x_{k+1}]$, etc
- One two-part basis function is defined at each non-boundary node
- Tails of two basis functions overlap at each non-boundary element



Concepts of "strong" and "weak" solutions

- Problem with linear basis functions: $u_i \in C(\Omega)$, but $u_i \notin C^2(\Omega)$, and $u_i \notin C^1(\Omega)$, so $U_h(\Omega) \not\subseteq U(\Omega)$ and $u_h \notin U(\Omega)$
- $u_h \in H_0^1(\Omega)$ – is called a *weak* solution, since it solves minimization problem for $F(u_h)$ on a subspace of $H_0^1(\Omega)$, which imposes weaker smoothness requirements than $C_0^2(\Omega)$. Solution of the differential equation on $C_0^2(\Omega)$ would be a *strong* solution, since the smoothness requirements are stronger. Nevertheless, a weak solution may be a very good approximation:



Numerical integration

- Having chosen basis functions $u_i(x)$, $i = 1, \ldots, n$ the next step is to evaluate the integrals:

$$s_{ij} = \int_0^1 u_i'(x) u_j'(x)\, dx, \quad \text{and} \quad f_i = \int_0^1 f(x) u_i(x)\, dx$$

- In FEM the integrals will be evaluated over elements

$$\int_{x_{k-1}}^{x_k} g(x)\, dx$$

- The general numerical approximation of an integral (*quadrature formula*) is

$$\int_{x_{k-1}}^{x_k} g(x)\, dx \approx \sum_{m=1}^{r} w_m g(x_m)$$

where: r – number of quadrature points, w_m – weights, x_m – quadrature points.

Basic integration rules

- *mid-point* rule

$$\int_{x_{k-1}}^{x_k} g(x)\,dx \approx (x_k - x_{k-1})g(x_{k-1/2})$$

- *trapezoidal* rule

$$\int_{x_{k-1}}^{x_k} g(x)\,dx \approx \frac{x_k - x_{k-1}}{2}\left[g(x_{k-1}) + g(x_k)\right]$$

- *Simpson's* rule

$$\int_{x_{k-1}}^{x_k} g(x)\,dx \approx \frac{x_k - x_{k-1}}{6}\left[g(x_{k-1}) + 4g(x_{k-1/2}) + g(x_k)\right]$$

Integration with linear basis functions: RHS

Since each $u_i(x)$ is nonzero only on two elements joined at x_i:

$$
\begin{aligned}
f_i &= \int_0^1 f(x)u_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)u_i(x)\,dx + \int_{x_i}^{x_{i+1}} f(x)u_i(x)\,dx \\
&= \int_{x_{i-1}}^{x_i} f(x)\frac{x - x_{i-1}}{x_i - x_{i-1}}\,dx + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1} - x}{x_{i+1} - x_i}\,dx \\
&\approx \frac{x_i - x_{i-1}}{2}\left[f(x_{i-1})\frac{x_{i-1} - x_{i-1}}{x_i - x_{i-1}} + f(x_i)\frac{x_i - x_{i-1}}{x_i - x_{i-1}}\right] \\
&\quad + \frac{x_{i+1} - x_i}{2}\left[f(x_i)\frac{x_{i+1} - x_i}{x_{i+1} - x_i} + f(x_{i+1})\frac{x_{i+1} - x_{i+1}}{x_{i+1} - x_i}\right] \\
&= \left[\frac{x_i - x_{i-1}}{2} + \frac{x_{i+1} - x_i}{2}\right] f(x_i) = \frac{x_{i+1} - x_{i-1}}{2} f(x_i) = \frac{h_i + h_{i+1}}{2} f(x_i)
\end{aligned}
$$

if approximated with trapezoidal rule

S-matrix: overlapping of linear basis functions

In 1D two basis functions and their derivatives overlap on two adjacent elements:

$$
\begin{aligned}
&\dots \\
s_{i,i-2} &= \int_0^1 u_i'(x) u_{i-2}'(x)\, dx = 0 \\
s_{i,i-1} &= \int_0^1 u_i'(x) u_{i-1}'(x)\, dx = \int_{x_{i-1}}^{x_i} u_i'(x) u_{i-1}'(x)\, dx \\
s_{i,i} &= \int_0^1 u_i'(x) u_i'(x)\, dx = \int_{x_{i-1}}^{x_{i+1}} (u_i'(x))^2\, dx \\
s_{i,i+1} &= \int_0^1 u_i'(x) u_{i+1}'(x)\, dx = \int_{x_i}^{x_{i+1}} u_i'(x) u_{i+1}'(x)\, dx \\
s_{i,i+2} &= \int_0^1 u_i'(x) u_{i+2}'(x)\, dx = 0 \\
&\dots
\end{aligned}
$$

S-matrix for linear basis functions (1)

$$
\begin{aligned}
s_{i,i-1} &= \int_0^1 u_i'(x) u_{i-1}'(x)\, dx = \int_{x_{i-1}}^{x_i} u_i'(x) u_{i-1}'(x)\, dx \\
&= \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)' \left(\frac{x_i - x}{x_i - x_{i-1}} \right)' dx \\
&= -\int_{x_{i-1}}^{x_i} \frac{1}{(x_i - x_{i-1})^2}\, dx = -\frac{1}{x_i - x_{i-1}} = -\frac{1}{h_i}
\end{aligned}
$$

S-matrix for linear basis functions (2)

$$
\begin{aligned}
s_{i,i} &= \int_0^1 u_i'(x) u_i'(x)\, dx = \int_{x_{i-1}}^{x_{i+1}} \left[u_i'(x)\right]^2\, dx \\
&= \int_{x_{i-1}}^{x_i} \left[\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)'\right]^2 dx + \int_{x_i}^{x_{i+1}} \left[\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)'\right]^2 dx \\
&= \int_{x_{i-1}}^{x_i} \frac{1}{(x_i - x_{i-1})^2}\, dx + \int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2}\, dx \\
&= \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i} = \frac{1}{h_i} + \frac{1}{h_{i+1}}
\end{aligned}
$$

S-matrix for linear basis functions (3)

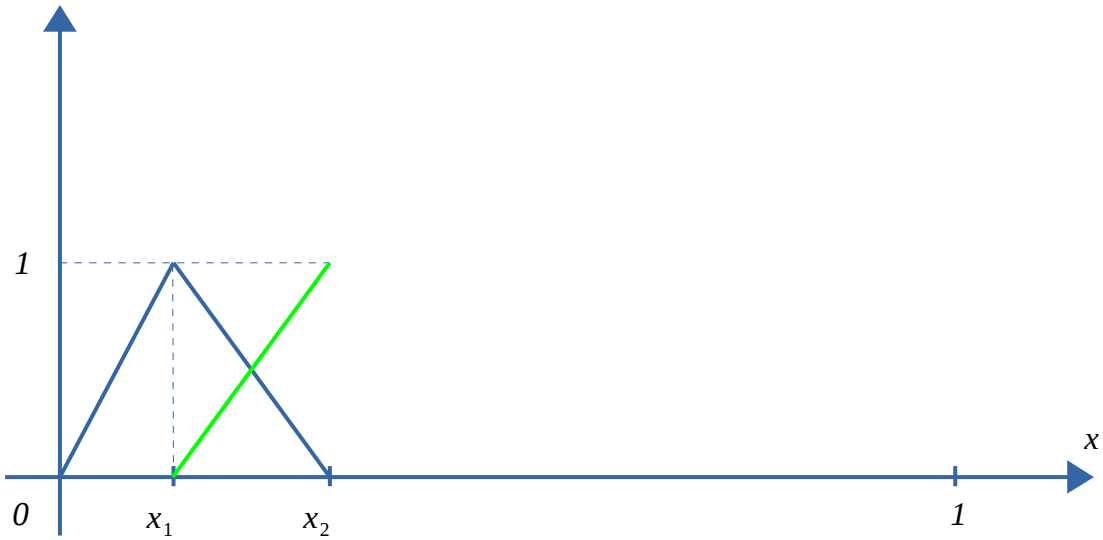
$$
\begin{aligned}
s_{i,i+1} &= \int_0^1 u_i'(x) u_{i+1}'(x)\, dx = \int_{x_i}^{x_{i+1}} u_i'(x) u_{i+1}'(x)\, dx \\
&= \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right)' \left(\frac{x - x_i}{x_{i+1} - x_i} \right)' dx \\
&= -\int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2}\, dx = -\frac{1}{x_{i+1} - x_i} = -\frac{1}{h_{i+1}}
\end{aligned}
$$

S-matrix: boundary elements (left)

$$\int_0^1 u_1'(x)u_0'(x)\,dx = 0$$

$$s_{1,1} = \int_0^1 u_1'(x)u_1'(x)\,dx = \frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} = \frac{1}{h_1} + \frac{1}{h_2}$$

$$s_{1,2} = \int_0^1 u_1'(x)u_2'(x)\,dx = -\frac{1}{x_2 - x_1} = -\frac{1}{h_2}$$

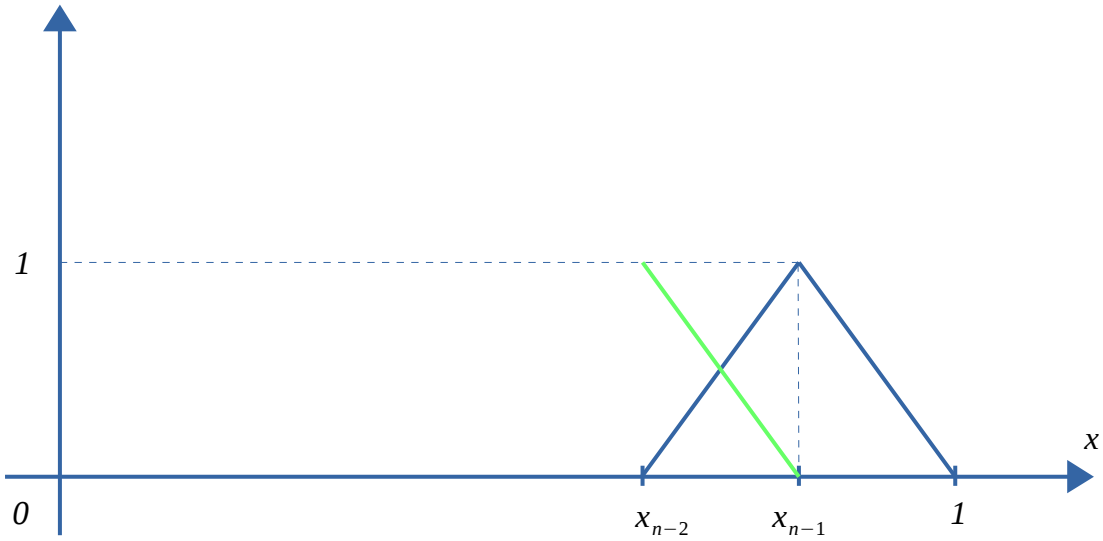


S-matrix: boundary elements (right)

$$s_{n-1,n-2} = \int_0^1 u'_{n-1}(x)u'_{n-2}(x)\,dx = -\frac{1}{x_{n-1}-x_{n-2}} = -\frac{1}{h_{n-1}}$$

$$s_{n-1,n-1} = \int_0^1 u'_{n-1}(x)u'_{n-1}(x)\,dx = \frac{1}{x_{n-1}-x_{n-2}} + \frac{1}{x_n - x_{n-1}} = \frac{1}{h_{n-1}} + \frac{1}{h_n}$$

$$\int_0^1 u'_{n-1}(x)u'_n(x)\,dx = 0$$



Ritz's FEM for 1D Poisson with zero Dirichlet BC's

On the uniform grid $h_i = h$, $i = 1, \ldots, n$:

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & & \ddots & & \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & & -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix} = h \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-2}) \\ f(x_{n-1}) \end{bmatrix}$$

Hence, FEM with linear basis functions on a uniform grid is equivalent to FDM if we identify $a_i = u(x_i)$.

Example: 1D Poisson with mixed homogeneous BC's

For the equation

$$-u'' = f, \quad x \in (0,1)$$
$$u(0) = 0, \quad u'(1) = 0$$

the functional is

$$F(u) = \int_0^1 \left[\frac{1}{2}(u')^2 - fu\right] dx$$

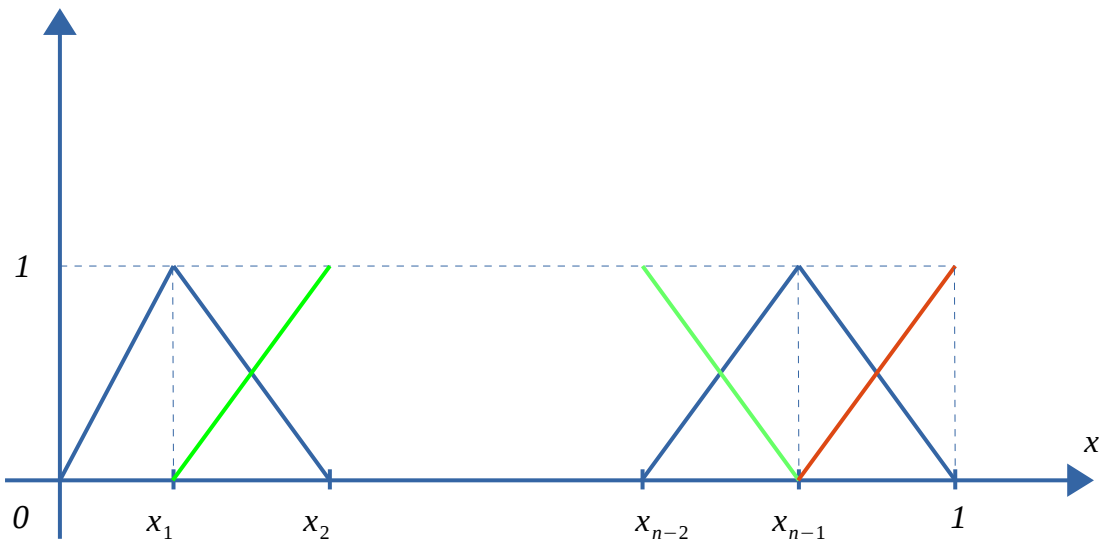
and $U(\Omega) = \{u \,|\, u'' \in C(\Omega);\ u(0) = 0\}$

Since the Neumann condition is homogeneous, there is no change in the functional with respect to Dirichlet case. However, the function space $U(\Omega)$ is different now, $u(1) = 0$ is no longer present as a condition.

Linear basis functions for the mixed BC's

The FEM approximation space $U_h(\Omega)$ can be constructed from linear basis functions that satisfy $u(0) = 0$ condition. However, since no such condition is enforced for $u(1)$, an additional linear basis function in the element $[x_{n-1}, x_n]$ is required leading to an additional unknown a_n.

$$u_n(x) = \frac{x - x_{n-1}}{x_n - x_{n-1}}, \quad x_{n-1} \leq x \leq x_n$$



Integration of $f(x)$ at the Neumann boundary

$$
\begin{aligned}
f_n &= \int_0^1 f(x)u_n(x)\,dx = \int_{x_{n-1}}^{x_n} f(x)u_n(x)\,dx = \int_{x_{n-1}}^{x_n} f(x)\frac{x - x_{n-1}}{x_n - x_{n-1}}\,dx \\
&\approx \frac{x_n - x_{n-1}}{2}\left[f(x_{n-1})\frac{x_{n-1} - x_{n-1}}{x_n - x_{n-1}} + f(x_n)\frac{x_n - x_{n-1}}{x_n - x_{n-1}}\right] \\
&= \frac{x_n - x_{n-1}}{2} f(x_n) = \frac{h_n}{2} f(x_n)
\end{aligned}
$$

if approximated with trapezoid rule

S-matrix at the Neumann boundary

$$s_{n-1,n-2} = \int_0^1 u'_{n-1}(x)u'_{n-2}(x)\,dx = -\frac{1}{x_{n-1}-x_{n-2}} = -\frac{1}{h_{n-1}}$$

$$s_{n-1,n-1} = \int_0^1 u'_{n-1}(x)u'_{n-1}(x)\,dx = \frac{1}{x_{n-1}-x_{n-2}} + \frac{1}{x_n - x_{n-1}} = \frac{1}{h_{n-1}} + \frac{1}{h_n}$$

$$s_{n-1,n} = \int_0^1 u'_{n-1}(x)u'_n(x)\,dx = -\frac{1}{x_n - x_{n-1}} = -\frac{1}{h_n}$$

$$s_{n,n-1} = \int_0^1 u'_n(x)u'_{n-1}(x)\,dx = -\frac{1}{x_n - x_{n-1}} = -\frac{1}{h_n}$$

$$s_{n,n} = \int_0^1 u'_n(x)u'_n(x)\,dx = \frac{1}{x_n - x_{n-1}} = \frac{1}{h_n}$$

FEM for 1D Poisson with mixed homogeneous BC's

On the uniform grid $h_i = h$, $i = 1, \ldots, n$:

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & & \ddots & & \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = h \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ f(x_n)/2 \end{bmatrix}$$

Example: 1D Poisson with non-homogeneous mixed BC's

For the equation

$$-u'' = f, \quad x \in (x_0, x_n)$$
$$u(x_0) = a, \quad -u'(x_n) = b$$

the functional is

$$F(u) = \int_{x_0}^{x_n} \left[\frac{1}{2}(u')^2 - fu \right] dx + u(x_n)b$$

and $U(\Omega) = \{u \,|\, u'' \in C(\Omega);\ u(x_0) = a\}$

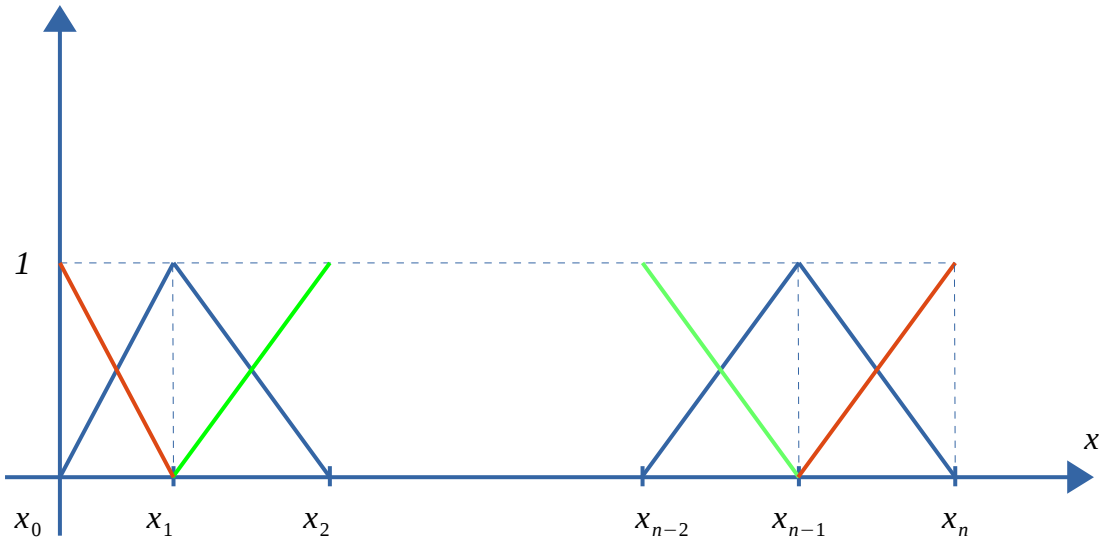
Since the Neumann condition is not homogeneous, there is an additional term in the functional. The function class $U(\Omega)$ is also different with respect to homogeneous BC's.
Dirichlet BC's are called *essential BC's*, since they always have to be included into $U(\Omega)$.

Linear basis functions for non-homogeneous BC's

Two linear basis functions are added at the nodes x_0 and x_n

$$u_0(x) = \frac{x_1 - x}{x_1 - x_0}, \quad x_0 \le x \le x_1$$

$$u_n(x) = \frac{x - x_{n-1}}{x_n - x_{n-1}}, \quad x_{n-1} \le x \le x_n$$



Incorporating essential Dirichlet BC's

All functions $u_h(x) \in U_h(\Omega)$ should satisfy essential Dirichlet BC's, i.e.,

$$U_h(\Omega) = \{u_h \,|\, u_h(x) = \sum_{i=0}^{n} a_i u_i(x);\, u_h(x_0) = a\}$$

This can be achieved by adding some basis functions $u_i(x)$ and fixing some of the coefficients a_i. In the present case one extra basis function $u_0(x)$ is sufficient and its coefficient needs to be fixed:

$$u_h(x) = \sum_{i=0}^{n} a_i u_i(x) = a_0 u_0(x) + \sum_{i=1}^{n} a_i u_i(x)$$

$$\text{Dirichlet BC: } u_h(x_0) = a \quad \Rightarrow$$

$$u_h(x_0) = a_0 u_0(x_0) + \sum_{i=1}^{n} a_i u_i(x_0) = a_0 + 0 = a \quad \Rightarrow \quad \boxed{a_0 = a}$$

$$u_h(x) = a u_0(x) + \sum_{i=1}^{n} a_i u_i(x)$$

Ritz's equations for coefficients

Substitute the representation of $u_h(x)$ in $F(u)$ and derive the Ritz equations ($j = 0$ is not required, since $a_0 = a$ is fixed)

$$\frac{\partial F(u_h)}{\partial a_j} = 0, \quad j = 1, \ldots, n; \quad u_h(x) = \sum_{i=0}^{n} a_i u_i(x)$$

$$\frac{\partial}{\partial a_j} \left\{ \int_{x_0}^{x_n} \left[\frac{1}{2} \left(\sum_{i=0}^{n} a_i u_i' \right)^2 - f \sum_{i=0}^{n} a_i u_i \right] dx + b \sum_{i=0}^{n} a_i u_i(x_n) \right\}$$

$$= \int_{x_0}^{x_n} \left[\frac{1}{2} 2 \sum_{i=0}^{n} a_i u_i' \frac{\partial}{\partial a_j} \sum_{i=0}^{n} a_i u_i' - f \frac{\partial}{\partial a_j} \sum_{i=0}^{n} a_i u_i \right] dx + b u_j(x_n)$$

$$= \sum_{i=0}^{n} a_i \int_{x_0}^{x_n} u_i' u_j' \, dx - \int_{x_0}^{x_n} f u_j \, dx + b u_j(x_n) = 0$$

Moving BC's to the right-hand side

$$\sum_{i=0}^{n} a_i \int_{x_0}^{x_n} u_i' u_j' \, dx - \int_{x_0}^{x_n} f u_j \, dx + b u_j(x_n) = 0$$

$$\sum_{i=1}^{n} a_i \int_{x_0}^{x_n} u_i' u_j' \, dx = \int_{x_0}^{x_n} f u_j \, dx - a \int_{x_0}^{x_n} u_0' u_j' \, dx - b u_j(x_n)$$

$$j = 1, \ldots, n$$

The Neumann term $b u_j(x_n)$ will be nonzero only for $j = n$, since $u_j(x_n) = 0$ for $j \neq n$.

Additional contributions to boundary elements

$$a \int_{x_0}^{x_n} u_0' u_j' \, dx, \quad j = 1, \ldots, n$$

$$a \int_{x_0}^{x_n} u_0' u_1' \, dx = a \int_{x_0}^{x_1} \left(\frac{x_1 - x}{x_1 - x_0} \right)' \left(\frac{x - x_0}{x_1 - x_0} \right)' \, dx$$

$$= a \int_{x_0}^{x_1} \frac{-1}{(x_1 - x_0)^2} \, dx = a \frac{-1}{x_1 - x_0} = -\frac{a}{h_1}$$

$$a \int_{x_0}^{x_n} u_0' u_j' \, dx = 0, \quad j = 2, \ldots, n$$

$$b u_n(x_n) = b \frac{x_n - x_{n-1}}{x_n - x_{n-1}} = b$$

FEM for 1D Poisson with non-homogeneous BC's

On the uniform grid $h_i = h$, $i = 1, \ldots, n$:

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & & \ddots & & \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = h \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ f(x_n)/2 \end{bmatrix} + \frac{1}{h} \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}$$